ARA

An Automatic Theorem Prover for Relation Algebras

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17.06.2000

Peirce (The Logic of Relatives, 1883):

> The logic of relatives is highly multiform; it is characterized by innumerable inferences, and by various distinct conclusions from the same sets of premises.
>
> (...)
>
> The effect of these pecularities is that this algebra cannot be subjected to hard and fast rules like those of the Boolian calculus; and all that can be done in this place is to give a general idea of the way of working with it.

RA, RRA, SA: Definitions

Relation Algebra (RA): Structure

$$\mathcal{A} = (A, +, ^-, \odot, \smile, \mathring{1})$$

satisfying the equations (BI)-(BX) for all $R, S, T \in A$.

Proper Relation Algebra: All elements of A are actually binary relations, operations are the usual set-theoretic ones.

Representable Relation Algebra (RRA): $\mathcal{A}$ representable iff. $\mathcal{A}$ isomorphic to a proper relation algebra.

Semi-associative Relation Algebra (SA): Associativity of $\odot$ (equation (BIV)) replaced by equation $R \odot 1 = (R \odot 1) \odot 1$.

Finite Variable Logic

n-**variable logic:** First order predicate logic with restricted language: only n distinct variables $x_0, \ldots, x_{n-1}$.

n-**variable calculus:** All axioms resp. rules restricted to n variables.

Cut-rule: In n-variable logic calculi with cut-rule are usually stronger than their counterparts without cut-rule (e.g. sequence calculus).

n-**variable resolution** is weaker than the n-variable sequence calculus with cut-rule.

Gordeev's Reduction Predicate Calculi: n-variable calculi without cut-rule (RPC_n), equivalent in proof power to n-variable sequence calculus with cut-rule (SCC_n).

The ARA Prover

- Prover for the RPC_n calculi.

- Front-end to convert RA equations to 3-variable sentences of first-order logic.

- Different reduction strategies, e.g.:

  ORP: oldest reduction possibility

  LP: complementary literal pair strategy

- Various additional simplification rules, e.g.:
  - Priority for shortening rules.
  - Subgoal generation.
  - Pure literal deletion.

Calculus of Relations

Basic objects: Binary relations $R, S, T, \ldots$

Basic operations:

complementation:	R^{-}	$\forall xy(xR^{-}y \;\Leftrightarrow\; \neg xRy)$
conversion:	$R^{\smile}$	$\forall xy(xR^{\smile}y \;\Leftrightarrow\; yRx)$
rel. multiplication:	$R \odot S$	$\forall xy(xR \odot Sy \;\Leftrightarrow\; \exists z(xRz \wedge zSy))$
abs. addition:	$R + S$	$\forall xy(xR + Sy \;\Leftrightarrow\; xRy \vee xSy)$
relative unit:	$\mathring{1}$	$\forall xy(x\mathring{1}y \;\Leftrightarrow\; x = y)$
relational equiv.:	$R = S$	$\forall xy(xRy \;\Leftrightarrow\; xSy)$
relational incl.:	$R \leq S$	$\forall xy(xRy \;\Rightarrow\; xSy)$

Derived operations:

relative addition:	$R \oplus S$	$R \oplus S \;=\; (R^{-} \odot S^{-})^{-}$
relative zero:	$\mathring{0}$	$\mathring{0} \;=\; \mathring{1}^{-}$

Calculus of Relations (cont.)

Examples:

$R \odot R \leq R$	transitivity	$\forall xy(\exists z(xRz \wedge zRy) \Rightarrow xRy)$
$R \leq R^{\smile}$	symmetry	$\forall xy(xRy \Rightarrow yRx)$
$R^{\smile} \odot R \leq \mathring{1}$	functionality	$\forall xyz(zRx \wedge zRy \Rightarrow x = y)$

Basic identities (Peirce 1883):

$$\mathring{1} \leq R \oplus R^{-\smile} \qquad R \odot R^{-\smile} \leq \mathring{0}$$

$$R \odot (S \oplus T) \leq (R \odot S) \oplus T \qquad R \oplus (S \odot T) \leq (R \oplus S) \odot T$$

$$R \odot S \leq T \;\Leftrightarrow\; R^{\smile} \odot T^{-} \leq S^{-} \;\Leftrightarrow\; T^{-} \odot S^{\smile} \leq R^{-}$$

Tarski's Axiomatization

$$
\begin{array}{lrcl}
\text{(BI)} & R + S & = & S + R \\
\text{(BII)} & R + (S + T) & = & (R + S) + T \\
\text{(BIII)} & (R^- + S)^- + (R^- + S^-)^- & = & R \\
\text{(BIV)} & R \odot (S \odot T) & = & (R \odot S) \odot T \\
\text{(BV)} & (R + S) \odot T & = & R \odot T + S \odot T \\
\text{(BVI)} & R \odot \mathring{1} & = & R \\
\text{(BVII)} & R^{\smile\smile} & = & R \\
\text{(BVIII)} & (R + S)^{\smile} & = & R^{\smile} + S^{\smile} \\
\text{(BIX)} & (R \odot S)^{\smile} & = & S^{\smile} \odot R^{\smile} \\
\text{(BX)} & R^{\smile} \odot (R \odot S)^- + S^- & = & S^-
\end{array}
$$

RPC_n **Rewrite Systems**

$$
\begin{array}{lrcl}
(R1) & A \vee \top & \longrightarrow & \top \\
(R2) & L \vee \neg L & \longrightarrow & \top \\
(R3) & A \wedge \top & \longrightarrow & A \\
(R4) & A \vee (B \wedge C) & \longrightarrow & (A \vee B) \wedge (A \vee C) \\
(R5) & \exists x A & \longrightarrow & \exists x A \vee A[x/t] \\
(R6) & \forall x A \vee B & \longrightarrow & \forall x A \vee B \ \vee \ \forall y(\forall y B \vee A[x{-}y][x/y]) \\
(R6') & \forall x A & \longrightarrow & \forall x A \vee A[-x]
\end{array}
$$

$A[-x]$ replaces all literals containing a free x by $\bot$, $A[y{-}x]$ only if $x \neq y$.

F is provable in RPC_n iff. it can be reduced to $\top$ in finitely many reduction steps.

$\vee$ and $\wedge$ are assumed to be associative and commutative.

Finite Variable Logic and Relation Algebras: The Link

Tarski (1941): Every sentence of relation algebra can be translated to a 3-variable logic sentence and vice versa.

Maddux (1978):

1. A sentence is valid in SA iff. its translation is provable in SCC_3.
2. A sentence is valid in RA iff. its translation is provable in SCC_4.
3. A sentence is valid in RRA iff. its translation is provable in SCC_ω.

Experimental Results

problem	source	class	strat.	proofs	steps	time
3.2(v)	[TG87]	SSA	LI	2	46	0.13
3.2(vi)	[TG87]	SSA	LI	2	41	0.10
3.2(xvii)	[TG87]	SSA	LI	3	22	0.04
3.2(xviii)	[TG87]	RA	AI	1	12	0.05
3.1(iii)(ϵ)	[TG87]	RA	AI	6	104	0.20
3.2(xix)	[TG87]	RA	LA	3	25	0.05
Thm 2.7	[CT51]	RA	AI	1	12	0.04
Thm 2.11	[CT51]	RA	AI	1	19	0.05
Cor 2.19	[CT51]	RA	AI	1	77	75.17
Dedekind	[DG98]	RA	AI	1	37	0.09
Cor 2.19	[CT51]	RRA	AI	1	38	0.14

Run-times in seconds on a Sun E450 running at 400 MHz.